# Current Laws and the Maxwell Equations

Robert S. Eisenberg

Bob.eisenberg@gmail.com

**Your DOI: 10.13140/RG.2.2.34171.63524**

Department of Applied Mathematics

Illinois Institute of Technology

Chicago IL USA

Department of Physiology and Biophysics

Rush University

Chicago IL USA

April 29, 2025

## Abstract

The Maxwell equations govern electrodynamics. But current laws—not the Maxwell equations—are used to design the circuits of our technology. Our goal is to derive and understand those current laws in the context of electrodynamics.

Current Laws can be derived by taking the divergence of the Maxwell Ampere Law $\mathbf{div\ curl\ B} = \mathbf{0} = \mathbf{div}\ (\mu_0(\mathbf{J} + \varepsilon_0\ \partial\mathbf{E}/\partial t)) = \mathbf{div}(\mu_0 \mathbf{J}_{total})$. The total current $\mathbf{J}_{total}$ forms a divergence free solenoidal field in which conduction and displacement current flows without sources or sinks, in loops and circuits in two dimensional systems. These loops enforce correlations on movements of individual charges on all time scales. The loops change the qualitative nature of electric fields including those in thermal motion in ways that cannot be calculated from Coulomb's law for uncorrelated charges and their movements. When captured in branching networks, these circuits form the devices of our technology. The classical Kirchhoff current law is a long-time approximation to the stationary behavior of $\mathbf{div}\ (\mathbf{J} + \varepsilon_0\ \partial\mathbf{E}/\partial t)$. When applied to classical (topological) models of circuits, Kirchhoff's law neglects the displacement current required to conform with the Maxwell equations. That displacement current is customarily provided by supplementing the topological network with poorly defined invented circuit elements called parasitic or stray capacitances. These elements are needed to be compatible with the Maxwell Ampere Law and to fit experimental data—even qualitatively. These invented elements provide a poor foundation for the design of circuits in which displacement currents are very large. Displacement currents are very large in the circuits of modern computers that switch in $10^{-10}$s. Invented circuit elements provide an uncertain and thus inadequate numerical foundation because the location and value of the invented elements are subjective, an art as much as a science. It seems obviously wise to have a treatment of networks that actually satisfies the laws of electrodynamics without subjectively chosen invented circuit elements. Continued use of the classical Kirchhoff Current Law eq. (5) far outside its domain of validity is a poor practice that undermines the mathematical basis of circuit analysis. Current laws should themselves be compatible with the laws of electrodynamics in a way that the classical Kirchhoff law is not.

The Maxwell equations govern electrodynamics. But different equations—current laws—are used to design the circuits of our technology. Our goal is to derive and understand those current laws in the context of electrodynamics.

Current Laws can be derived from the Maxwell Ampere Law [1-5]

$$\mathbf{curl\,B} = \mu_0(\mathbf{J} + \varepsilon_0\,\partial\mathbf{E}/\partial t) = \mu_0\mathbf{J}_{total} \tag{1}$$

Here, $\mathbf{B}$ is the magnetic field, $\mu_0$ is the magnetic constant, $\mathbf{E}$ is the electric field, $\varepsilon_0$ is the electric constant, $t$ is time, and $\mathbf{J}$ is the flux of any charge with mass, in electrical units, no matter what the energy source of the flux. $\mathbf{J}$ includes dielectric and polarization currents no matter how brief and transient, no matter what fields drive them.

Equation (1) requires a separate description of how charge moves and charge density is changed by applied electric and magnetic fields, in phenomena loosely called polarizability when they are reversible. See the devastating criticism in [2], p.500-507 that shows the classical treatment of polarizability $\mathbf{P}$ is not unique in a modern atomic scale context. The classical treatment is thus untenable in my view and should be taught only as a historical relic, not an essential part of the field equations of electrodynamics. A similar opinion is in Feynman [5], section 13-4. See extensive discussion in [3], and section 4.3.4 of [1].

Polarization can be treated rather like the description of compressibility of mass is treated in theories of velocity fields in fluid mechanics [6]: a phenomenological treatment summarizing relevant experiments is needed that extends to all the experimental conditions in which the electrodynamics of polarization will be computed. When little is known concerning polarization [7], it is customary and appropriate to use a single dielectric constant.

$\mathbf{J}$ includes irreversible charge movements that require descriptions not included in the Maxwell equations. The drift diffusion theory of charge carriers in semiconductors [8, 9] is one important example central to much of our technology; another is the closely related PNP [10] (Poisson Nernst Planck) treatment of ions in electrolytes and channels. $\mathbf{J}$ also includes charge movements driven by temperature or pressure gradients and convection [11-13].

If we take the divergence of both sides of eq.(1), we can apply an identity true for any field $\widetilde{\mathbf{B}}$**,** not just the magnetic field,

$$\mathbf{div\ curl}\,\widetilde{\mathbf{B}} = 0 \tag{2}$$

We then derive what I call the Maxwell Current Law, since it is true whenever and wherever the Maxwell-Ampere law eq.(1) is true. Note that the Maxwell Current Law does not depend on any property of matter. It is as universal and precise as the Maxwell Ampere Law or the closely related laws of special relativity [1], Ch. 12.

**Maxwell Current Law** $$\mathbf{div}(\,\mathbf{J} + \varepsilon_0\,\partial\mathbf{E}/\partial t\,) = \mathbf{div}\,\mathbf{J}_{total} = \mathbf{0} \tag{3}$$

Divergence-free fields like those of eq.(3) are often called 'solenoidal'. They describe the velocity fields of incompressible fluids and have special properties which we discuss below.

Maxwell [14] felt that $\mathbf{J}_{total}$ was very important, "one of the chief peculiarities of [his] treatise". In his Vol. 2, Section 610, p. 232, Maxwell used unusually judgmental language. He called $\mathbf{J}_{total}$ the "true electric current" when he wrote [15] "One of the chief peculiarities of this treatise is the doctrine which it asserts, that the true electric current, that on which the electromagnetic phenomena depend, is not the same thing as the current of conduction, but that the time-variation of the electric displacement, must be taken into account in estimating the total movement of electricity, so that we must write, [the sum] .... as an equation of true currents." It is useful to read Maxwell's more concise formulation [16] that is easily written in the modern language. Of course, the discussion in this paper is based on the mathematical properties of Maxwell's equation (3), not his subjective judgements of its importance.

Solenoidal fields have special properties [17]. They have no sources or sinks, in the ordinary sense of the word. Their physical origin is in dipoles which gives them quite different properties from fields like conduction current $\mathbf{J}$ with their typical 'monopole' sources and sinks.

> The special properties of solenoidal fields, with dipolar origin, are extensively described in three dimensions in the context of magnetic fields $\mathbf{B}$ in [18, 19]. But the physical meaning of the dipolar origin of total current $\mathbf{I}_{total}$ needs much more study particularly in the finite

two dimensional systems in which electric circuits are embedded. The important special case $\mathbf{B} = 0$ is described in [20]. But the physical meaning of zero total current $\mathbf{I}_{total} = 0$ is quite different from $\mathbf{B} = 0$. Zero magnetic field cannot be excluded in many real problems. Zero total current is a more degenerate case that may sometimes be excluded from consideration, as can currents flowing to infinity, particularly in finite systems.

In finite neighborhoods [1], and two dimensional systems ([17] p. 29), the flow lines (of $\mathbf{J}_{total}$) are loops. In those cases, conduction current $\mathbf{J}$ combines with displacement current $\varepsilon_0\, \partial\mathbf{E}/\partial t$ to form circuits that loop back on themselves as they flow from one place to another. Flows in solenoidal fields form circuits in two-dimensional systems, and also in "finite neighborhoods, but not globally", in three-dimensional systems, p. 242 [1].

The loop nature of total current in two dimensional systems reflects the properties of the Maxwell Ampere law and as such is as universal and exact as any physical law. The loop nature of total current (in two dimensional systems) enforces strong correlations in the movements and concentrations of charges on all time scales and in all conditions. These correlations produce behavior very different from the uncorrelated motions assumed in most treatments of thermal noise [21], and in most simulations of molecular dynamics.

Time scales are important here: The Maxwell equations successfully describe experiments over an enormous time range, including those of explosions, even the time scale of radiation implosion in nuclear weapons. Thus, Maxwell's Current Law (3) is true on the time scale of thermal noise, or chemical reactions. The correlations implied by its solenoidal nature are not damped out by friction, they are not affected by the 'inertia', on the fastest or slowest time scales. Indeed, they are likely to be dominant sources of complex motion of charges whether atomic or macroscopic. They cannot be understood by analysis of individual uncorrelated motions using Coulomb's law.

The correlations enforced by the Maxwell Current Law (3) form an additional constraint on a chemical reaction, no matter how fast the reaction. This constraint is in addition to the constraints provided by the equation of the chemical reaction (e.g., mass action), no matter how fast the reaction is. The $k$ reactants of a chemical reaction accumulate their individual fluxes $\mathbf{J}_k$, as shown in the combination of eq. (6) and the law of mass action, but the total current $\mathbf{J}_{total}$. carried by the reactants does not accumulate, ever. The interactions of these constraints has been worked out in detail in a mathematically consistent model of the important enzyme cytochrome c oxidase [22].

The correlations enforced by the Maxwell Current Law (3) form an additional restraint on thermal motion. For example, in a simple series circuit, eq. (3) implies that total current is independent of location, at any time [23]. Thus, for this special case, the partial derivatives of total current with respect to location are zero. The law for total current acts as a ***perfect*** low pass spatial filter! The spatial noise of total current does not depend on location at all within any one branch of a circuit. The temporal noise varies wildly in (nearly) Brownian motion. The correlations enforced by the Maxwell Current Law are so strong that the spatial total current does not vary at all.

**Engineering Circuits.** Engineers try to confine the inherent loop currents (of two-dimensional solenoidal fields of total current) to branched one-dimensional networks where Current Laws are used to describe them. The loop currents of these networks can follow simple laws like Kirchhoff's Current Law eq.(5) in special cases, i.e., at long times. Kirchhoff's current law has been used to design circuits successfully for nearly two centuries, without even using differential equations, let alone computerized numerical analysis.

Current laws are of great practical importance. They allow engineers to design devices that are used extensively in modern life. Most of our electronic devices and technology are implemented by those branched networks. The loop currents of these networks allow design of essentially all our complex electronic devices, digital, analog, linear and nonlinear, many of which perform as designed in $10^{-10}$s. It is remarkable that circuit laws originally designed in the 1850's to describe one second signals in telegraphs also are useful 175 years later for analyzing $10^{-10}$s signals in our computers.

Total current $\mathbf{J}_{total}$ flows in three dimensions in the ground planes of the integrated circuits of our computers [24]. These ground planes occupy large fractions of the space in actual integrated circuits. Current in ground planes has not been analyzed with the Maxwell Current Law as far as I know.

The properties of total current in two-dimensional branching networks allow some important analysis without knowing the details of each $\mathbf{J}_k$ just as many properties of electronic circuits can be understood without detailed understanding of the physics of each circuit element using the 'art of electronics' [25].

Consider the simplest circuit, a series combination of circuit elements and wires that form a single branch of a more general network. Maxwell's Current Law (3) says that the total current is exactly equal everywhere in that branch at every time, equal in every wire and every circuit element, whatever the microphysics of the element or wire. As Landauer put it [26] on p. 112: "It is, after all, the sum of electron current and displacement current which has no divergence. One of those two components can take over from the other."

It is important here to be precise, to avoid confusion. Here, 'Circuit element' for a resistor is defined by **R,** a single positive real number, and the resistor is defined by Ohm's law, without associated corrections for nonideal or capacitive or inductive behavior. Networks are branched one dimensional circuits embedded in a finite two-dimensional space. 'Network' and 'circuit' are defined only by the connection (topology) of circuit elements. Classical (topological) networks do not depend on the dimensions or other details of the actual layout of circuit elements in the device. Topological networks and circuits include only actual circuit elements by convention: they do not include supplementary elements like stray or parasitic capacitances or the nonideal properties of conductors, even their inductance.

When total current is confined into a branching network, one has a law for the current in each branch. The branch current is $\mathbf{J}_{total}$ integrated over the area of each wire and element of the network.

***Maxwell Current Law for Networks: Sum of total currents flowing into a node is zero.*** (4)

Mathematically, it is clear that some of the properties of three-dimensional $\mathbf{J}_{total}$ cannot be found in a branching network of a finite number of components. Singularities in three-dimensional space are an example. Complexities in solenoidal flow lines associated mostly with $\mathbf{J}_{total} = 0$ are another [20]. The question of what three-dimensional properties can be preserved in finite two-dimensional branching networks of circuits is interesting mathematics, but it need not concern us here.

What is clear is that engineers can confine total current to branching two-dimensional networks well enough to design functional circuits of great importance over an enormous range of distances and times that do not require understanding of the detailed physics of the components of the circuits. Indeed, they have done so for nearly two centuries, starting with telegraph networks spanning oceans, long before computers were available to analyze current flow on micro let alone nanosecond time scales.

Engineers design their circuits using the Kirchhoff Current Law for $\mathbf{J}$.

***Kirchhoff Current Law: Sum of the conduction currents $J$ flowing into a node is zero.*** (5)

Kirchhoff's Current law uses the stationary properties of total current in eq. (4) & (3) as an approximation to the time dependent currents $\varepsilon_0\, \partial\mathbf{E}/\partial t$ of the Maxwell Ampere Law (1) discussed in general by Maxwell [14]. It is obviously a special case of the Maxwell Current Law when displacement current $\varepsilon_0\, \partial\mathbf{E}/\partial t = 0$.

When displacement current is not zero, the Kirchhoff current law (5) needs to be supplemented by invented circuit elements to simulate the universal displacement current $\varepsilon_0\, \partial\mathbf{E}/\partial t$. These fictional elements are called stray or parasitic capacitances. Fictional elements are also used to approximate other properties of the real circuit (as found on circuit boards or in integrated circuits) that are not present in topological networks.

Time dependent properties are prominent in experiments on modern time scales and so treatments of circuits that use the classical Kirchhoff laws without explicitly including parasitic capacitances do not describe the qualitative properties observed from modern circuits. Classical treatments are bewildering to those doing these experiments when they do not conform with what is observed. When confronted with unexpected transients, newcomers are likely to abandon use of the current laws altogether, confronted with its qualitative difference from reality, until they are initiated in the art of practical electronic design [25]. They certainly are not likely to recognize the underlying principles that are universal and exact and are embodied in the Maxwell Ampere and Maxwell Current Laws. They are unlikely to try to apply current laws to complex biological systems like myelinated nerve fibers or mitochondria.

It is important to make quantitative estimates, so the reader understands the scale of the failure of the Kirchhoff Current Law when applied to modern circuits, if used without invented circuit elements. The Kirchhoff Current Law eq. (5) adequately describes circuits with resistance **R** only at times when the displacement current is negligible, namely at times much longer than **RC** where C is roughly 10 picofarads [27, 28]. This means that the ***Kirchhoff Current Law eq. (5) is an inadequate description of current flow in modern computer circuits*** that switch in nanoseconds or faster. Modern computer circuits switch on time scales several orders of magnitude quicker than the range of validity of Kirchhoff's current law.

The classical Kirchhoff law eq. (5) says conduction current **J** does not accumulate. But eq. (3) implies that conduction current does accumulate.

$$\mathbf{div\,J} = -\mathbf{div}(\varepsilon_0\,\partial\mathbf{E}/\partial t) \tag{6}$$

Current accumulation (as net charge) is obvious when observing the qualitative properties of modern circuit in contradiction to the classical Kirchhoff law eq. (5).

The classical continuity equation of electrodynamics shows the same behavior [1, 3], see [27], particularly Section 6-10, p. 288-9. To say this crucial point again for emphasis: conduction currents like electron currents do not behave as described by the Kirchhoff Current Law (5) in modern circuits in which time dependence is usually apparent and important.

Charge accumulation is also involved when a ***sequence*** of responses are studied. Sequences of individual 'voltage clamp' pulses, are the bread-and-butter of classical membrane conductances of biological systems [29] and the ion channels [30-32] that produce those conductances. As one moves from one member of the sequence to the next, experimental conditions change. (That is why a sequence is used!) As experimental conditions change between each member of the sequence, potential changes and charge almost certainly accumulates. This

***charge accumulation appears in each one of the sequence of measurements even if the equations of each response of the circuit are stationary and do not explicitly include time.*** As Maxwell implied (*op. cit.* [15]) one must ***always*** deal with “the true electric current". If the displacement current $\varepsilon_0\,\partial\mathbf{E}/\partial t$ does not appear explicitly in an apparently stationary measurement, it is still present in the initial conditions of each measurement.

The classical Kirchhoff Current Law eq. (5) is routinely modified by engineers, so it becomes a much better description of practical circuits as we have stated. Engineers 'pad out' the idealized topological circuit by inventing circuit elements that introduce the displacement current $\varepsilon_0\,\partial\mathbf{E}/\partial t$ and also approximate the nonideal properties of real circuits. These fictional capacitances are added to the circuit in the widely used network analysis program by LTSpice [33, 34] using the names *cshunt* and *cshuntin*. Artfully assigned, these invented circuit elements allow Kirchhoff's Current Law eq. (5) to describe the qualitative properties of the circuits of our computers that switch far faster than **RC**.

It should clearly be understood, however, that the location and value assigned to these invented circuit elements are subjective and part of the art of circuit design [25] much more than the science [25]. Stationary approximations like Kirchhoff’s eq. (5) combined with fictional circuit elements produce confusion and uncertainty particularly when they assume $\varepsilon_0\,\partial\mathbf{E}/\partial t = \mathbf{0}$ for circuits that switch in $10^{-10}$s. And of course they are mathematically unjustified. They cannot provide a solid or unique numerical basis for design because they cannot be objectively measured. Different scientists will choose different locations and values for the fictional elements. Circuit design then becomes unnecessarily subjective, as recognized in the very title of the encylopedic textbook “Art of Electronics” [25].

Art is not science. And art does not always succeed. One of the goals of science is to replace art with computation when possible. Subjective circuit elements do not provide a ***reproducible*** numerical foundation for design. An objective foundation is important because we need the billions of circuits in our computers to switch reliably hundreds of millions of times each second in whatever setting we put them in.

## Conclusion

It seems obviously wise to have a treatment of networks that actually satisfies the laws of electrodynamics without subjective fictional invented circuit elements. It is a simple matter to modify analysis of topological networks made with the classical Kirchhoff current law to include the universal displacement current $\varepsilon_0\, \partial \mathbf{E}/\partial t$. Topological networks are often further modified to provide more realistic descriptions of real integrated circuits by adding different types of fictional circuit elements.

Continued use of the classical Kirchhoff Current Law eq. (5) far outside its domain of validity is a poor practice that is hard to justify. It undermines the mathematical basis of circuit design and is likely to produce confusion. Current laws should themselves be compatible with the laws of electrodynamics in a way that the classical Kirchhoff Current Law is not.

## Acknowledgement

It is a pleasure to thank Ardyth Eisenberg for her contributions to this paper, and for all she has brought to my life.

## References


1. Griffiths, D. J. 2024. Introduction to electrodynamics. Fifth Edition. Cambridge University Press.
2. Purcell, E. M., and D. J. Morin. 2013. Electricity and magnetism. Cambridge University Press.
3. Zangwill, A. 2013. Modern Electrodynamics. Cambridge University Press, New York.
4. Maxwell, J. C. 1865. VIII. A dynamical theory of the electromagnetic field. Philosophical transactions of the Royal Society of London(155):459-512.
5. Feynman, R. P., R. B. Leighton, and M. Sands. 1963. The Feynman: Lectures on Physics, Vol. 2. Mainly Electromagnetism and Matter. Addison-Wesley Publishing Co., also at http://www.feynmanlectures.caltech.edu/II_toc.html, New York.
6. Eisenberg, B., C. Liu, and Y. Wang. 2022. On Variational Principles for Polarization Responses in Electromechanical Systems. Communications in Mathematical Sciences 20(6):1541-1550.
7. Eisenberg, R. S. 2019. Dielectric Dilemma. preprint available at https://arxiv.org/abs/1901.10805.
8. Vasileska, D., S. M. Goodnick, and G. Klimeck. 2010. Computational Electronics: Semiclassical and Quantum Device Modeling and Simulation. CRC Press, New York.
9. Selberherr, S. 1984. Analysis and Simulation of Semiconductor Devices. Springer-Verlag, New York.
10. Eisenberg, R. 2019. PNP what is in a name july 25-1 2019. pdf 10.31224/osf.io/2739d. engrXiv. August 3.
11. Eisenberg, B., Y. Hyon, and C. Liu. 2010. Energy Variational Analysis EnVarA of Ions in Water and Channels: Field Theory for Primitive Models of Complex Ionic Fluids. Journal of Chemical Physics 133(10):104104.
12. Wang, Y., and C. Liu. 2022. Some Recent Advances in Energetic Variational Approaches. Entropy 24(5):721.
13. Xu, S., P. Sheng, and C. Liu. 2014. An energetic variational approach for ion transport. arXiv preprint arXiv:1408.4114.
14. Maxwell, J. C. 1865. A Treatise on Electricity and Magnetism (reprinted 1954). Dover Publications, New York.
15. Eisenberg, R. S. 2024. Maxwell's True Current; see arXiv 2309.05667. Computation 12(2):22-46 see arXiv 2309.05667.
16. Maxwell, J. C. 1865. A Dynamical Theory of the Electromagnetic Field. Philosophical Transactions of the Royal Society of London 155:459-512.
17. Chorin, A. J., J. E. Marsden, and J. E. Marsden. 1990. A mathematical introduction to fluid mechanics. Springer.
18. Blums, E., A. Cebers, and M. M. Maiorov. 2010. Magnetic fluids. Walter de Gruyter.
19. Rosensweig, R. E. 2013. Ferrohydrodynamics. Courier Corporation.

20. Franklin, J., D. Griffiths, and D. Schroeter. 2024. A taxonomy of magnetostatic field lines. American Journal of Physics 92(8):583-592.

21. Eisenberg, B. 2006. The value of Einstein's mistakes. *Letter to the Editor:* "Einstein should be allowed his mistakes …". Physics Today 59(4):12.

22. Xu, S., R. Eisenberg, Z. Song, and H. Huang. 2023. Mathematical Model for Chemical Reactions in Electrolytes Applied to Cytochrome c Oxidase: An Electro-Osmotic Approach. Computation 11(12):253.

23. Eisenberg, R. S. 2020. Electrodynamics Correlates Knock-on and Knock-off: Current is Spatially Uniform in Ion Channels. Preprint on arXiv at https://arxiv.org/abs/2002.09012.

24. Joffe, E. B., and K.-S. Lock. 2010. Grounds for Grounding. Wiley-IEEE Press, NY.

25. Horowitz, P., and W. Hill. 2015. The Art of Electronics. Cambridge University Press.

26. Landauer, R. 1992. Conductance from transmission: common sense points. Physica Scripta 1992(T42):110.

27. Ulaby, F. T., and U. Ravaioli. 2015. Fundamentals of Applied Electromagnetics. Pearson.

28. Ulaby, F. T., and M. M. Maharbiz. 2010. Circuits. NTS press.

29. Huxley, A. 1972. The Quantitative Analysis of Excitation and Conduction in Nerve. From Nobel Lectures, Physiology or Medicine 1963–1970. Elsevier, Amsterdam.

30. Neher, E. 1997. Ion channels for communication between and within cells Nobel Lecture, December 9, 1991. Nobel Lectures, Physiology or Medicine 1991-1995. N. Ringertz, editor. World Scientific Publishing Co, Singapore, pp. 10-25.

31. Sakmann, B., and E. Neher. 1995. Single Channel Recording. Plenum, New York.

32. Zheng, J., and M. C. Trudeau. 2015. Handbook of ion channels. CRC Press.

33. Analog_Devices. 2025. LTspice IV Getting Started Guide. (posted).

34. Brocard, G. 2013. The LTspice IV simulator: manual, methods and applications. Würth Elektronik.